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On the value of Reversionary Annuities payable half-yearly, quarterly, &c., according to the conditions which prevail in practice. *By* THOMAS BOND SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 26th April, 1869.]

THE ordinary formula for the value of a reversionary annuity, $a_x - a_{xy}$, is not strictly applicable to the cases which occur in practice. For it assumes that the first payment of the annuity is made at the end of the year in which y dies, and the last payment at the end of the year before that in which x dies. But the conditions which prevail in practice are different; for whenever a reversionary annuity is granted, it will run from the death of y, and the first payment will be made twelve months after that death, *i.e.*, on the average, six months later than is assumed in the above formula. On the other hand, the reversionary annuity will in practice be payable up to the day of the death of x; or, on the average, a full year's payment will be made six months later than the last payment is made according to the above formula. Hence we may say that in the reversionary annuity of practice, exactly the same payments are made, on the average, as in the theoretical annuity; but that each of them is made six months later in the one case than in the other; and it follows that the value of the reversionary annuity of practice is, approximately,

$$\frac{a_x - a_{xy}}{\sqrt{1+i}}, \text{ or } (a_x - a_{xy})\left(1 - \frac{i}{2} + \frac{3}{8}i^2 - \frac{5}{16}i^3 + \ldots\right) \ldots (1)$$

Let us now look at the question from a rather different point of view. We may say approximately that in consequence of the reversionary annuity running from the death of y only, and not from the end of the previous year, the value is diminished by that of £½ payable on the death of y, in the lifetime of x, or by $\frac{1}{2}A_{\overset{\;\;1}{\overline{x,y}}}$.

Again, we may say in the same way that in consequence of the annuity being payable up to the death of x, instead of ceasing at the end of the previous year, its value is increased by the value of £½ payable on the death of x after the death of y, or by $\frac{1}{2}A_{\overset{2\;\;}{\overline{x,y}}}$.

Hence the correction to be applied to the ordinary formula will be

$$\frac{1}{2}A_{\overset{2\;\;}{\overline{x,y}}} - \frac{1}{2}A_{\overset{\;\;1}{\overline{x,y}}}.$$

But since $A_{\overset{2\;\;}{\overline{x,y}}} = A_x - A_{\overset{1\;\;}{\overline{x,y}}}$, this becomes

$$\begin{aligned} &\frac{1}{2}\left(A_x - A_{\overset{1\;\;}{\overline{x,y}}} - A_{\overset{\;\;1}{\overline{x,y}}}\right) \\ &= \frac{1}{2}(A_x - A_{xy}) \\ &= -\frac{1}{2}\frac{i(a_x - a_{xy})}{1+i}\,; \end{aligned}$$

and the value of the reversionary annuity of practice is approximately

$$\begin{aligned} &(a_x - a_{xy})\left(1 - \frac{1}{2}\cdot\frac{i}{1+i}\right) \\ &= (a_x - a_{xy})\left(1 - \frac{i}{2} + \frac{i^2}{2} - \text{\&c.}\right) \ldots. \quad (2) \end{aligned}$$

which agrees with the former formula as far as the two first terms of the second factor. The conditions here supposed do not agree so closely with those of practice as those previously assumed; and the formula (1) is therefore to be considered preferable.

Next, suppose that the reversionary annuity, instead of being paid yearly, is payable half-yearly; then it is clear that exactly the same amount will be paid to the annuitant as before; but in each year there will be two payments of £½, instead of a single payment of £1. Now the value of £1 due in a year, is $\frac{1}{1+i}$; and the value of the two payments of £½, at the end of half a year and a year respectively, is $\frac{1}{2\sqrt{1+i}} + \frac{1}{2(1+i)}$. The latter value bears to the former a ratio equal to $\frac{\sqrt{1+i}+1}{2}$.

This being true of each year's payments, the value of the reversionary annuity payable half-yearly will be greater than that of the same annuity payable yearly in the same ratio; and will therefore be approximately

$$\begin{aligned} &\frac{a_x - a_{xy}}{\sqrt{1+i}}\cdot\frac{\sqrt{1+i}+1}{2} \\ &= (a_x - a_{xy})\cdot\frac{1 + (1+i)^{-\frac{1}{2}}}{2} \\ &= (a_x - a_{xy})\left(1 - \frac{i}{4} + \frac{3}{16}i^2 - \frac{5}{32}i^3 + \ldots.\right) \end{aligned}$$

Next, if the reversionary annuity is payable m times a year, the value at the beginning of a year of the payments £$\frac{1}{m}$, to be made at m equal intervals during the year is

$$\frac{1}{m}\left(v^{\frac{1}{m}}+v^{\frac{2}{m}}+\;\dots\;+v\right)$$

$$=\frac{v^{\frac{1}{m}}}{m}\,\frac{1-v}{1-v^{\frac{1}{m}}}\,.$$

and the ratio which this bears to the value of £1 to be paid at the end of the year is

$$\frac{v^{\frac{1}{m}}}{m}\cdot\frac{1-v}{1-v^{\frac{1}{m}}}\cdot\frac{1}{v}=\frac{1}{m}\cdot\frac{i}{v^{-\frac{1}{m}}-1}$$

$$=\frac{1}{m}\cdot\frac{i}{(1+i)^{\frac{1}{m}}-1}$$

$$=1+\frac{i}{2}\left(1-\frac{1}{m}\right)-\frac{i^2}{12}\left(1-\frac{1}{m^2}\right)+\frac{i^3}{24}\left(1-\frac{1}{m^2}\right)\cdot\;\dots$$

as I have shown in vol. xiii., p. 206.

Hence the value of the reversionary annuity payable m times a year is approximately

$$\frac{a_x-a_{xy}}{\sqrt{1+i}}\left\{1+\frac{i}{2}\left(1-\frac{1}{m}\right)-\frac{i^2}{12}\left(1-\frac{1}{m^2}\right)+\frac{i^3}{24}\left(1-\frac{1}{m^2}\right)\dots\right\}\;..\;(3)$$

Here making $m=2$, we get the value of the annuity when payable half-yearly

$$\frac{a_x-a_{xy}}{\sqrt{1+i}}\left\{1+\frac{i}{4}-\frac{i^2}{16}+\frac{i^3}{32}-\;\dots\right\}$$

$$=(a_x-a_{xy})\left(1-\frac{i}{4}+\frac{3}{16}i^2-\frac{5}{32}i^3+\;\dots\right)\;\dots\;(4)$$

as found above.

Again, making $m=4$ the value becomes

$$\frac{a_x-a_{xy}}{\sqrt{1+i}}\left\{1+\frac{3}{8}i-\frac{5}{64}i^2+\frac{5}{128}i^3-\;\dots\right\}$$

$$=(a_x-a_{xy})\left\{1-\frac{i}{8}+\frac{7}{64}i^2-\frac{3}{32}i^3+\;\dots\right\}\;\dots\;(5)$$

which is the approximate value of the reversionary annuity when payable quarterly.

Making m infinite, the value of the annuity supposed to be payable momently, becomes

$$\frac{a_x-a_{xy}}{\sqrt{1+i}}\left(1+\frac{i}{2}-\frac{i^2}{12}+\frac{i^3}{24}-\ldots\ldots\right)$$

$$=(a_x-a_{xy})\left(1+\frac{i^2}{8}+\frac{5}{24}i^3\ldots\ldots\right)\ldots\ldots\ldots\ (6)$$

The treatment of this problem by former writers does not appear to have been happy. In a paper by the late Dr. Young, reprinted in this *Journal* (vol. vii., p. 22), it is stated that since the same correction, viz. $\frac{1}{4}$, is applied to each of the terms in a_x-a_{xy} when the annuity is payable half-yearly, we get the paradoxical result that the value of the reversionary annuity is the same whether payable yearly or half-yearly. His explanation of the paradox is by no means clear; and it appears to be impossible to conclude from his remarks what he believed to be the correct value of either the yearly or half-yearly reversionary annuity.

The following explanation will, I believe, be found satisfactory. The reversionary annuity of which the value is a_x-a_{xy}, runs, not from the death of y, but from the end of the previous year, so that a full year's payment is supposed to be made to x at the end of the year in which y dies. If, however, the annuity be payable half-yearly, then x will receive only $\frac{1}{2}$ at the end of the year in which y dies; and the other $\frac{1}{2}$ will be payable in the middle of that year, and will be received by y, should he be then alive, the chance of which, on the supposition of equal decrements, is $\frac{1}{2}$. Thus, by the alteration to half-yearly payments, x has an even chance of losing $\frac{1}{2}$; or, taking the average, may be said to lose $\frac{1}{4}$ payable in the middle of the year in which y dies; and this loss just counterbalances the advantage (also equal to $\frac{1}{4}$, when referred to the same epoch) he derives in consequence of the payments in the succeeding years being made half-yearly. The value of the annuity thus remains unaltered under the suppositions made.

In Vol. iv. p. 299, of this *Journal*, the following formula was given by Mr. Holmes Ivory, for the value of a reversionary annuity payable half-yearly for the life of x after the death of y;

$$a_x-a_{xy}-\frac{1}{4}\mathrm{A}_{\overset{1}{xy}}\ldots\ldots\ldots\ldots\ldots\ (7)$$

The reasoning by which this formula was arrived at does not seem to be satisfactory; and the result appears to be fairly open to the charge brought against it by Mr. Thomas Carr (Vol. vii. p. 110), that it represents the annuity when payable half-yearly as costing *less* than when payable yearly. It will be found that the formula gives approximately the value of a reversionary annuity payabl half-yearly, which runs from the death of y, but *ceases with the last half-yearly payment prior to the death of x*, not being continued to the day of the death of x. For, as we have seen above, the value of the reversionary annuity payable yearly, which runs from the death of y, is

$$a_x - a_{xy} - \frac{1}{2} A_{\overset{1}{\overline{xy}}} \dots\dots\dots\dots\dots (8)$$

this annuity not being continued to the death of x, but ceasing with the last yearly payment which precedes that death. If now this annuity is made payable half-yearly, its value at the time it is entered upon is increased by $\frac{1}{4}$; and consequently its value at the present time is increased by $\frac{1}{4} A_{\overset{1}{\overline{xy}}}$. Adding this to (8) we get Mr. Ivory's formula; but there is no reason to suppose that he had in view the annuity here described. If any further proof were required of the accuracy of the views here expressed, it would be furnished by the fact that the formula given by Mr. Ivory for the value of a reversionary annuity payable quarterly, viz.:

$$a_x - a_{xy} - \frac{3}{8} A_{\overset{1}{\overline{xy}}} \dots\dots\dots\dots (9)$$

does not admit of any similar interpretation. For it is easily seen that the value of the reversionary annuity, payable quarterly, which runs from the death of y, but ceases with the last quarterly payment prior to the death of x, is

$$\left(a_x - a_{xy} - \frac{1}{2} A_{\overset{1}{\overline{x.y}}}\right) + \frac{3}{8} A_{\overset{1}{\overline{x.y}}}$$

$$= a_x - a_{xy} - \frac{1}{8} A_{\overset{1}{\overline{x.y}}}.$$

It will be seen that Mr. Ivory's formulæ represent the value of the annuity when payable quarterly as being less than when payable half-yearly.

Not having succeeded in fully understanding the drift of Mr. Carr's remarks, I will make no comment on them, but simply add that he, like Dr. Young, has omitted to give any formulæ for the value of reversionary annuities payable half-yearly, quarterly, &c.

It will be seen that the reasoning by which the formulæ (1) and (3) were established, applies equally to the values of reversionary annuities purchased as an investment. If the value of such an annuity payable yearly during the life of x after the death of y—the first payment being made at the end of the year in which y dies, and the last at the end of the year before that in which x dies—be a; then the value of such an annuity running from the death of y to the death of x is approximately

$$\frac{a}{\sqrt{1+i}}=a\left(1-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ldots\ldots\right)\ldots\ldots\ldots\ldots\qquad(10)$$

If, furthermore, the annuity is payable half-yearly, its value will be

$$a\left(1-\frac{i}{4}+\frac{3}{16}i^2-\frac{5}{32}i^3+\ \ldots\ldots\right);\ \ldots\ldots\qquad(11)$$

and if quarterly,

$$a\left(1-\frac{i}{8}+\frac{7}{64}i^2-\frac{3}{32}i^3+\ \ldots\ldots\right)\ldots\ldots\qquad(12)$$

These formulæ are applicable in whatever way we determine a. We may either with Mr. Jellicoe take it equal to

$$\frac{1}{\mathrm{P}_x+d_5}-1-(a_{xy})_{3\frac{1}{2}}\ \ldots\ldots\ldots\ldots\qquad(13)$$

or we may take either of the two formulæ I have explained in my paper *On the value of Reversionary Life Interests,* in the Number of the *Journal* for January last. It will be noticed however that we must not, under the suppositions now made, subtract $\frac{1}{2}\mathrm{A}_{\overset{1}{\overline{xy}}}$, as I have there done, so that instead of the formulæ there given we shall have for a the formulæ

$$\frac{1}{\mathrm{P}_x+d_6}-1-(a_{xy})_6\ \ldots\ldots\ldots\ldots\qquad(14)$$

and

$$\frac{1-(\mathrm{P}_x+d_6)(1+a_{xy})_6}{\mathrm{P}_x+d_5}\ \ldots\ldots\ldots\ldots\qquad(15)$$

As a first approximation to the truth, which will often be quite sufficient, we may say that interest for half a year is to be deducted from these values when the annuity is payable yearly; and interest for one quarter, and eighth part of a year respectively, when the annuity is payable half-yearly and quarterly; this interest being of course computed at six per cent.

In a similar manner we may deduce the approximate value of *any complete annuity* payable half-yearly or quarterly from the value of the same annuity payable yearly. We have only to multiply this value by

$$1+\frac{i}{4}-\frac{i^2}{16}+\frac{i^3}{32}-\ \ldots\ldots$$

and

$$1+\frac{3i}{8}-\frac{5i^2}{64}+\frac{5i^3}{128}-\ \ldots\ldots$$

respectively, or simply to add interest for a quarter of a year and three-eighths of a year respectively.

This simple rule applies to all annuities, whether whole life, temporary, deferred, or intercepted; and whether on one life or on several; provided only that the annuity is *complete*, or a proportionate part is payable up to the day of the death of the life or lives on which the annuity depends. For, if that is the case, it is very easy to see that whether the annuity be payable yearly, half-yearly, quarterly, or more frequently, the sum of the payments to the annuitant will in all cases be the same, the increase in the value arising solely from the interest on the sums prepaid.

This rule, however, is not strictly correct, even on the supposition of equal decrements. Comparing the cases of yearly and half-yearly payments, we see that in the latter case the annuitant gains half a year's interest on £½ for every year through which the nominee lives. But in the year in which the nominee dies, there is only an even chance of the annuitant obtaining any advantage, since if death occurs in the first half of the year, there is no advantage from the half-yearly payment of the annuity; and the advantage when death does occur in the second half of the year is limited to an average gain of three months' interest on £½. The value given by the rule is therefore too large, and the error is approximately equal to

$$\left\{\frac{\sqrt{1+i}-1}{2}-\frac{(1+i)^{\frac{1}{4}}-1}{4}\right\}\mathrm{A}_x$$

$$=\left\{\frac{3}{16}i-\frac{5}{128}i^2+\frac{9}{512}i^3-\ .\ .\ .\ .\right\}\mathrm{A}_x.$$

The error will therefore generally affect the third place of decimals, unless A_x be very small; but usually not the second place of decimals.

I will conclude this paper by finding an expression for the value of the annuity here considered, on the supposition of equal decrements, *i.e.* on the supposition that the deaths are distributed uniformly over each year of life.

First suppose that the annuity is payable yearly during the life of x on the anniversary of the death of y. Then considering the instant distant from the present time, $n+t$, where n is an integer and $t<1$; there will be a payment of the annuity due at that instant if (1) x be then alive and (2) y died at the time t in any one of the 1st, 2nd . . . nth years. Now the probability of x being alive at the time $n+t$, is, on the supposition of equal decrements,

$$(1-t)p_{x,\,n}+tp_{x,\,n+1}$$

Also the probabilities of y having died at the time t in the 1st, 2nd, . . . nth, years respectively, are

$$
\begin{array}{l}
(1-p_{y,1})dt \\
(p_{y,1}-p_{y,2})dt \\
(p_{y,2}-p_{y,3})dt \\
\vdots \\
(p_{y,n-1}-p_{y,n})dt
\end{array}
$$

And the sum of these, *i.e.* the probability of y having died at the time t in some one of the n years is

$$(1-p_{y,n})dt.$$

It is, in fact, obvious that on the suppositions we have made, assuming y to have died within the n years, he is as likely to have died at any time t (<1) as at any other such time. Hence the probability of a payment of the annuity falling due at the time $n+t$, is

$$\{(1-t)p_{x,n}+tp_{x,n+1}\}(1-p_{y,n})dt;$$

and the present value of the payment so made will be found by multiplying this quantity by v^{n+t}, and will therefore be

$$v^{n+t}\{(1-t)p_{x,n}+tp_{x,n+1}-(1-t)p_{xy,n}-tp_xp_{x+1.y,n}\}dt.$$

The value of the annuity will now be found by integrating this expression with respect to t between the limits 0 and 1, and summing it with respect to n for the values 1, 2, 3 . . . to the end of life; and it is immaterial which of these operations we perform first.

Integrating first with respect to t, we have

$$\int v^t dt=\frac{v^t}{\log_e v}+\text{C}. \quad \therefore \int_0^1 v^t dt=-\frac{1-v}{\log_e v}=\frac{1-v}{\delta}=\frac{i}{(1+i)\delta},$$

putting δ for $-\log_e v$.
Again

$$\int_0^1 tv^t dt=\frac{1}{\delta^2}-\frac{v}{\delta}-\frac{v}{\delta^2} \text{ (vol. xiii., p. 359)}$$

$$=\frac{i-\delta}{(1+i)\delta^2},$$

whence $$\int_0^1 (1-t)v^t dt=\frac{i\delta-i+\delta}{(1+i)\delta^2}.$$

Thus the result of the integration becomes

$$v^n\frac{i\delta-i+\delta}{(1+i)\delta^2}(p_{x,n}-p_{xy,n})+v^n\frac{i-\delta}{(1+i)\delta^2}(p_{x,n+1}-p_xp_{x+1.y,n}).$$

Next summing this with respect to n, for the values 1, 2, 3 . . . to the end of life, we have

$$\Sigma v^n p_{x,n}=a_x; \qquad \Sigma v^n p_{xy,n}=a_{xy};$$

$$\Sigma v^n p_{x,n+1}=\frac{a_x}{v}-p_x; \qquad \Sigma v^n p_{x+1.y,n}=a_{x+1.y}.$$

So that the result of the summation is

$$\frac{i\delta-i+\delta}{(1+i)\delta^2}(a_x-a_{xy})+\frac{i-\delta}{(1+i)\delta^2}\{(1+i)a_x-p_x-p_xa_{x+1.y}\}$$

$$=\frac{i^2}{(1+i)\delta^2}a_x-\frac{i\delta-i+\delta}{(1+i)\delta^2}a_{xy}-\frac{i-\delta}{\delta^2}\cdot\frac{a_{x.y-1}}{p_{y-1}} \quad . . . . (16)$$

since $$a_{x.y-1}=\frac{p_xp_{y-1}}{1+i}(1+a_{x+1.y}).$$

To test this formula, suppose that the life y is replaced by certainty; then p_{y-1} becomes 1; and a_{xy}, $a_{x.y-1}$ both become a_x; and the formula becomes

$$\frac{i^2-i\delta+i-\delta-(1+i)(i-\delta)}{(1+i)\delta^2}\cdot a_x, \text{ which } =0.$$

This result is clearly correct, as in the case supposed the reversionary annuity is indefinitely postponed and will never be entered upon.

Again, suppose the life x replaced by certainty; then a_x becomes $v+v^2+$ . . . *ad inf.* $=\frac{v}{1-v}=\frac{1}{i}$. Also a_{xy} is replaced by a_y; and $\frac{a_{x.y-1}}{p_{y-1}}$ by $\frac{a_{y-1}}{p_{y-1}}$, which is equal to $\frac{1+a_y}{1+i}$. Hence the formula becomes

$$\frac{i}{(1+i)\delta^2}-\frac{i\delta-i+\delta}{(1+i)\delta^2}\cdot a_y-\frac{i-\delta}{(1+i)\delta^2}(1+a_y)$$

$$=\frac{1}{(1+i)\delta}-\frac{i}{(1+i)\delta}\cdot a_y$$

$$=\frac{1-ia_y}{(1+i)\delta}.$$

Now by the suppositions we have made, the reversionary annuity has been replaced by a perpetual annuity certain running from the death of y, the value of which is that of $\frac{1}{i}$ payable at the instant of y's death or $\frac{\overline{A}_y}{i}$. But on the supposition of equal decrements we have (as is shown by Baily)

$$\bar{A}_y = \frac{i}{\delta}A_y = \frac{i}{\delta}\frac{1-ia_y}{1+i}.$$

So that the value of the reversionary annuity certain is $\frac{1-ia_y}{(1+i)\delta}$, which agrees with the value already found.

Let us next consider the case when the annuity is "complete"; and investigate the value of the addition that is to be made to (16) on account of the payment up to the day of death. Suppose that y dies in some year at the time t, which, as before, is <1; then the annuity is payable at the time t in each subsequent year during the life of x; and when x dies, a proportionate part is payable from the date of the last payment to the death of x. Suppose that x dies at the time τ, reckoned (like t) from the beginning of the year; then there are two cases according as τ is $>$ or $<$ t. If $\tau > t$, or x die after the payment of the annuity has been made in any year, there will be $\tau - t$ payable; but if $\tau < t$, or x die before the payment of the annuity is made, there will be $1 - t + \tau$ payable.

First suppose $\tau > t$; then the value of the correction, supposing that x dies in the nth year, and therefore at the time $n-1+\tau$ reckoned from the present time, is

$$(\tau - t)v^{n-1+\tau}(p_{x,n-1} - p_{x,n})(1 - p_{y,n})dt\,d\tau$$

This has to be integrated with respect to τ, between the limits t and 1; then it has to be integrated with respect to t between the limits 0 and 1; and lastly it has to be summed with respect to n giving it the values 1, 2, . . . to the end of life. Now

$$\int(\tau - t)v^{\tau}d\tau = -\frac{(\tau - t)v^{\tau}}{\delta} + \int\frac{v^{\tau}}{\delta}d\tau$$

$$= -\frac{\tau - t}{\delta}\cdot v^{\tau} - \frac{v^{\tau}}{\delta^2} + \mathrm{C}$$

$$\therefore \qquad \int_t^1 (\tau - t)v^{\tau}d\tau = \frac{v^t}{\delta^2} - \frac{1-t}{\delta}v - \frac{v}{\delta^2} = \mathrm{H}, \text{ suppose.}$$

Next integrating with respect to t

$$\int \mathrm{H}dt = -\frac{v^t}{\delta^3} - \left(t - \frac{t^2}{2}\right)\frac{v}{\delta} - \frac{vt}{\delta^2} + \mathrm{C}$$

$$\therefore \qquad \int_0^1 \mathrm{H}dt = \frac{1}{\delta^3} - \frac{v}{\delta^3} - \frac{v}{2\delta} - \frac{v}{\delta^2}$$

$$= \frac{i}{(1+i)\delta^3} - \frac{1}{2(1+i)\delta} - \frac{1}{(1+i)\delta^2}$$

$$= \frac{2i - \delta^2 - 2\delta}{2(1+i)\delta^3}.$$

This has now to be multiplied into

$$\begin{aligned}&\Sigma v^{n-1}(p_{x,n-1}-p_{x,n})(1-p_{y,n})\\ &=\Sigma v^{n-1}\left(p_{x,n-1}-p_{x,n}-\frac{p_{x-1\cdot y,n}}{p_{x-1}}+p_{xy,n}\right)\\ &=1+a_x-\frac{a_x}{v}-\frac{a_{x-1\cdot y}}{vp_{x-1}}+\frac{a_{xy}}{v}\\ &=1-ia_x-(1+i)\left(\frac{a_{x-1\cdot y}}{p_{x-1}}-a_{xy}\right).\end{aligned}$$

Thus the value of the correction when $\tau > t$ becomes

$$\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}\left\{1-ia_x-(1+i)\left(\frac{a_{x-1\cdot y}}{p_{x-1}}-a_{xy}\right)\right\} \quad . \; . \; . \; (17)$$

Next suppose $\tau < t$; then the value of the correction on the supposition that x dies in the $(n+1)$th year is

$$(1-t+\tau)v^{n+\tau}(p_{x,n}-p_{x,n+1})(1-p_{y,n})dt\,d\tau.$$

This has to be integrated with respect to τ between 0 and t; then to be integrated with respect to t between 0 and 1; and lastly to be summed with respect to n giving it the values 1, 2, ... to the end of life.

First, $$\int(1-t+\tau)v^\tau d\tau=-(1-t+\tau)\frac{v^\tau}{\delta}+\int\frac{v^\tau d\tau}{\delta}$$
$$=-(1-t+\tau)\frac{v^\tau}{\delta}-\frac{v^\tau}{\delta^2}+\text{C}$$

$$\therefore \quad \int_0^t(1-t+\tau)v^\tau d\tau=\frac{1-t}{\delta}+\frac{1}{\delta^2}-\frac{v^t}{\delta}-\frac{v^t}{\delta^2}=\text{K, suppose.}$$

Next integrating with respect to t

$$\int \text{K}dt=\frac{1}{\delta}\left(t-\frac{t^2}{2}\right)+\frac{t}{\delta^2}+\frac{v^t}{\delta^2}+\frac{v^t}{\delta^3}+\text{C}$$

$$\begin{aligned}\therefore \quad \int_0^1\text{K}dt&=\frac{1}{2\delta}+\frac{1}{\delta^2}+\frac{v}{\delta^2}+\frac{v}{\delta^3}-\frac{1}{\delta^2}-\frac{1}{\delta^3}\\ &=\frac{1}{2\delta}+\frac{1}{(1+i)\delta^2}-\frac{i}{(1+i)\delta^3}\\ &=\frac{(1+i)\delta^2+2\delta-2i}{2(1+i)\delta^3}.\end{aligned}$$

And this has to be multiplied into

$$\begin{aligned}&\Sigma v^n(p_{x,n}-p_{x,n+1})(1-p_{y,n})\\ &=\Sigma v^n(p_{x,n}-p_{x,n+1}-p_{xy,n}+p_xp_{x+1\cdot y,n})\\ &=a_x-\left(\frac{a_x}{v}-p_x\right)-a_{xy}+p_xa_{x+1\cdot y}\\ &=-ia_x-a_{xy}+(1+i)\,\frac{a_{x\cdot y-1}}{p_{y-1}};\end{aligned}$$

so that the value of the correction when $\tau < t$ is

$$\frac{(1+i)\delta^2+2\delta-2i}{2(1+i)\delta^3}\left\{-ia_x-a_{xy}+(1+i)\frac{a_{x\cdot y-1}}{p_{y-1}}\right\} \quad \ldots\ldots (18)$$

The total value of the correction is therefore, adding together (17) and (18),

$$\begin{aligned}&\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}\left\{1-ia_x-(1+i)\frac{a_{x-1\cdot y}}{p_{x-1}}+(1+i)a_{xy}\right\}\\ &\qquad+\frac{(1+i)\delta^2+2\delta-2i}{2(1+i)\delta^3}\left\{-ia_x-a_{xy}+(1+i)\frac{a_{x\cdot y-1}}{p_{y-1}}\right\}\\ &=\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}-\frac{i^2}{2(1+i)\delta}\cdot a_x+\frac{(2i-\delta^2-2\delta)(2+i)-i\delta^2}{2(1+i)\delta^3}\cdot a_{xy}\\ &\quad-\frac{2i-\delta^2-2\delta}{2\delta^3}\cdot\frac{a_{x-1\cdot y}}{p_{x-1}}+\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3}\cdot\frac{a_{x\cdot y-1}}{p_{y-1}} \quad \ldots (19)\end{aligned}$$

To test this formula, suppose x replaced by certainty, then we get

$$\begin{aligned}&\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}-\frac{i}{2(1+i)\delta}+\frac{(2i-\delta^2-2\delta)(2+i)-i\delta^2}{2(1+i)\delta^3}\cdot a_y\\ &\qquad-\frac{2i-\delta^2-2\delta}{2\delta^3}\cdot a_y+\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3(1+i)}(1+a_y)\end{aligned}$$

which vanishes identically, as it should.

Next suppose y replaced by certainty, then we get

$$\begin{aligned}&\frac{2i-\delta^2-2\delta}{2(1+i)\delta^3}-\frac{i^2}{2(1+i)\delta}\cdot a_x+\frac{(2i-\delta^2-2\delta)(2+i)-i\delta^2}{2(1+i)\delta^3}\cdot a_x\\ &\qquad-\frac{2i-\delta^2-2\delta}{2\delta^3(1+i)}(1+a_x)+\frac{(1+i)\delta^2+2\delta-2i}{2\delta^3}\cdot a_x\end{aligned}$$

which also vanishes identically, as it should.

Adding (19) to (16) we should get a formula for the value of the complete reversionary annuity payable yearly; but this is so complicated as to be practically useless; and no good purpose would be gained by investigating the corresponding formulæ for the values of the annuity when payable half-yearly, &c.

By a process similar to the above it may be shown that the value of a contingent assurance payable at the instant of the death of x provided y survive, or $\overline{\mathrm{A}}^{1}_{xy}$, is, on the supposition of equal decrements,

$$\frac{i\delta-i+\delta}{(1+i)\delta^2}+\frac{2i\delta-2i+2\delta-i^2}{(1+i)\delta^2}a_{xy}-\frac{i\delta-i+\delta}{\delta^2}\cdot\frac{a_{x\cdot y-1}}{p_{y-1}}+\frac{i-\delta}{\delta^2}\cdot\frac{a_{x-1\cdot y}}{p_{x-1}} \quad \ldots (20)$$

Also that the value of an assurance payable at the instant of the death of the first of the two joint lives x and y, or $\overline{\mathrm{A}}_{xy}$, is, on the same supposition,

$$2\frac{i\delta-i+\delta}{(1+i)\delta^2}+2\frac{2i\delta-2i+2\delta-i^2}{(1+i)\delta^2}a_{xy}+\frac{2i-2\delta-i\delta}{\delta^2}\left(\frac{a_{x-1\cdot y}}{p_{x-1}}+\frac{a_{x\cdot y-1}}{p_{y-1}}\right).\;.\;.\quad(21)$$

On the Value of Reversionary Life Interests. By the late GRIFFITH DAVIES, ESQ., F.R.S. *Communicated by* GRIFFITH DAVIES, ESQ., *Actuary of the Law Life Assurance Society.*

TO find the present value of a Survivorship Annuity of £1 on the life of A to be entered on at the death of B, supposing the purchaser to effect an increasing insurance on A's life so that the sum insured may correspond with the amount of his outlay (of purchase money, premiums, and interest) at the end of each year up to the average duration of the joint lives of A and B, and to be at the end of that period of the exact amount which the annuity purchased would cover the annual interest thereon, beside providing for the premiums to keep it insured during the remainder of A's life.

Let $\mathrm{S}=$ sum advanced as purchase money for proposed annuity.
$\mathrm{P}=$ equated annual premium for requisite insurance.
$p=$ annual premium for the insurance of £1 on A's life.
$e=$ expectation of joint lives.
$i=$ rate of interest allowed the purchaser.
$r=1+i=$ amount of £1.
$v=\frac{1}{r}=$ present worth of £1.
$d=1-v$.

D, N, M, &c., as in Table XIII. (Davies' "Tables of Life Contingencies," published in 1825).

Then it is manifest that the improved amount of the purchase money at the end of x years $=\mathrm{S}r^x$; the improved amount of the premiums at the end of the same period $=\frac{\mathrm{P}r(r^x-1)}{i}$, and consequently the sum of both which then constitutes the aggregate outlay of the purchaser $=\mathrm{S}r^x+\frac{\mathrm{P}r(r^x-1)}{i}$, and if in this expression we substitute e for x we have the amount of the purchaser's outlay at the end of the period which A and B may be expected to